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Continuity of Positive Functionals on Topological $*$-Algebras

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1. INTRODUCTION

The automatic continuity of positive linear functionals (plfs) on Banach $*$-algebras possessing a bounded approximate identity was first proved by Varopoulos [20] using an extension of Cohen's factorization theorem [5]. Subsequently, it was shown (see [6, 12, 15]) that, for Fréchet $*$-algebras with uniformly bounded approximate identities, all of the ingredients of Varopoulos' proof remain valid; hence, every plf on such an algebra is automatically continuous. In this paper, several automatic continuity results for plfs on more general topological $*$-algebras (e.g., LMC $*$-algebras) without identities are established. Further, applications of these results to abstract harmonic analysis on certain locally compact groups are presented.

2. DEFINITIONS AND PRELIMINARIES

DEFINITION 2.1. A *(locally convex) topological $*$-algebra* A is an algebra with involution $x \to x^*$ which is a complex (locally convex) Hausdorff-topological vector space such that the ring multiplication is separately continuous and the involution is continuous.

DEFINITION 2.2. An algebra A is said to *factor* if $A = A^2$, where $A^2 = \operatorname{span}\{xy\colon x, y \in A\}$.

DEFINITION 2.3. A locally convex topological $*$-algebra A is an LMC *(locally multiplicatively convex) $*$-algebra* if its topology is given by a family $\{p_\lambda : \lambda \in \Lambda\}$ of *submultiplicative* (i.e., $p_\lambda(xy) \leqslant p_\lambda(x)p_\lambda(y)$) *symmetric* (i.e., $p_\lambda(x^*) = p_\lambda(x)$) seminorms. A complete metrizable LMC $*$-algebra is called a *Fréchet $*$-algebra.*

Remarks 2.4. (a) If A is an LMC $*$-algebra or a complete metrizable

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topological $*$-algebra, then the ring multiplication is automatically jointly continuous. (b) If A is a barreled locally convex topological $*$-algebra, then the ring multiplication is automatically hypocontinuous.

Notation 2.5. Let A be an LMC $*$-algebra with a defining family $\{p_\lambda : \lambda \in \Lambda\}$ of submultiplicative symmetric seminorms. For each index λ, the set $N_\lambda = \{x \in A : p_\lambda(x) = 0\}$ is a closed two-sided ideal in A, and the quotient $A_\lambda = A/N_\lambda$ is a normed $*$-algebra with norm $\|x_\lambda\| = p_\lambda(x)$ (x any preimage of x_λ under the natural $*$-homomorphism $\pi_\lambda : A \to A/N_\lambda$). The completion B_λ of A_λ is a Banach $*$-algebra with isometric involution. Two indices $\lambda, \mu \in \Lambda$ are related $\lambda < \mu$ if $p_\lambda(x) \leqslant p_\mu(x)$, for all $x \in A$. Whenever $\lambda < \mu$, there is a norm-decreasing $*$-homomorphism $\pi_{\lambda\mu} : A_\mu \to A_\lambda$ (defined by $\pi_{\lambda\mu}(\pi_\mu(x)) = \pi_\lambda(x)$); of course, $\pi_{\lambda\mu}$ extends uniquely to a norm-decreasing $*$-homomorphism (still denoted by $\pi_{\lambda\mu}$) from B_μ into B_λ.

THEOREM 2.6 [3, Theorems 2.4, 3.1]. *Let A be a complete* LMC *$*$-algebra with a defining family $\{p_\lambda : \lambda \in \Lambda\}$ of submultiplicative-symmetric seminorms, and let $\{B_\lambda : \lambda \in \Lambda\}$ and $\{\pi_{\lambda\mu} : \lambda < \mu\}$ be the associated families of Banach $*$-algebras and norm-decreasing $*$-homomorphisms as in* Notation 2.5. *Then $A = \operatorname{proj\,lim} B_\lambda$ (the projective limit of the $\{B_\lambda\}$); hence, in particular, if $x(\lambda) \in B_\lambda$ for all $\lambda \in \Lambda$ and if $\pi_{\lambda\mu}(x(\mu)) = x(\lambda)$ whenever $\lambda < \mu$, then there exists an $x \in A$ such that $x_\lambda = \pi_\lambda(x) = x(\lambda)$, for all $\lambda \in \Lambda$.*

DEFINITION 2.7. Let A be an LMC algebra without identity. For $x, y \in A$, let $x \circ y = x + y - xy$. An element $x \in A$ is said to be *quasi-regular* in A if there exists $y \in A$ such that $x \circ y = 0 = y \circ x$; y is called the *quasi-inverse* of x. The algebra A is a *Q-algebra* if the set of quasi-regular elements of A is open in A [13, App. E]. If $x \in A$, an LMC algebra without identity, then its *spectrum* is $\operatorname{Sp}(x, A) = \{c \in C \backslash \{0\} : c^{-1}x \text{ is not quasi-regular in } A\} \cup \{0\}$, and its *spectral radius* is $\mathrm{r}(x, A) = \sup\{|c| : c \in \operatorname{Sp}(x, A)\}$. The spectrum of each element of an LMC algebra is nonempty.

THEOREM 2.8 [13, Corollary 5.3; 12, Theorem 1.6]. *Let A be a sequentially complete* LMC *$*$-algebra with a defining family $\{p_\lambda : \lambda \in \Lambda\}$ of submultiplicative symmetric seminorms and let $\{B_\lambda : \lambda \in \Lambda\}$ be the associated family of Banach $*$-algebras as in* Notation 2.5. *Then, for each x in A*:

(a) $r(x, A) = \sup\{\mathrm{r}(x_\lambda, B_\lambda) : \lambda \in \Lambda\} = \sup_\lambda \lim_{n \to \infty} \sqrt[n]{p_\lambda(x_\lambda^n)}$,

(b) $\mathrm{r}(x^*, A) = \mathrm{r}(x, A)$,

(c) $\mathrm{r}(cx, A) = |c|\, \mathrm{r}(x, A)$, $c \in C$,

and, if x, y in A commute, then

(d) $\mathrm{r}(x + y, A) \leqslant \mathrm{r}(x, A) + \mathrm{r}(y, A)$,

(e) $\mathrm{r}(xy, A) \leqslant \mathrm{r}(x, A)\, r(y, A)$.

DEFINITION 2.9. Let A be an LMC algebra. An element $x \in A$ will be called *spectrally bounded* if $r(x, A) < +\infty$. Further, A will be termed (*almost*) *spectrally bounded* if the spectrally bounded elements of A are (dense in) all of A.

Remark 2.10. It is known that every Q-algebra is spectrally bounded, and further examples and facts relating to almost spectrally bounded and spectrally bounded LMC algebras are given in [13, Sec. 13].

DEFINITION 2.10. A net $\{e_\omega : \omega \in \Omega\}$ in a topological algebra A is called a *left* (resp., *right*, *two-sided*) *approximate identity* (lai) (resp., rai, tai) if $e_\omega x \to x$ (resp., $xe_\omega \to x$, $e_\omega x \to x$, and $xe_\omega \to x$), for every $x \in A$. It is said to be *pointwise operator bounded* (pob) if $\{e_\omega x : \omega \in \Omega\}$ (resp., $\{xe_\omega : \omega \in \Omega\}$, $\{e_\omega x : \omega \in \Omega\} \cup \{xe_\omega : \omega \in \Omega\}$) are bounded subsets of A, for every $x \in A$. It is *central* if $e_\omega x \times xe_\omega$, for every $\omega \in \Omega$, $x \in A$.

Remarks 2.12. (a) A *sequential* ai $\{e_\omega : \omega \in \Omega\}$ (here $\Omega = N$, the positive integers with their natural order) is always pob. (b) If A is a topological $*$-algebra and $\{e_\omega : \omega \in \Omega\}$ is an lai (rai), then $\{e_\omega^* : \omega \in \Omega\}$ is a rai (lai).

DEFINITION 2.13. A linear functional f on a topological $*$-algebra A is a *positive linear functional* (plf) if $f(x^*x) \geqslant 0$, for all $x \in A$, and is *central* if $f(xy) = f(yx)$, for all $x \in A$. A central pf on A will be called a *trace*.

DEFINITION 2.14. A plf f on a topological $*$-algebra A has *finite variation* if there exists a positive constant K such that $|f(x)|^2 \leqslant Kf(x^*x)$, for every $x \in A$; the infimum of all such K is called the *variation of f*, denoted $v(f)$. A plf f on a topological $*$-algebra A is *Hermitian* if $f(x^*) = \overline{f(x)}$ ($\overline{}$ denotes complex conjugation), for all $x \in A$, and *extendable* if it is Hermitian and has finite variation.

Notation 2.15. Let f be a plf on a topological $*$-algebra A and let $x \in A$. Define f_x on A by $f_x(y) = f(x^*yx)$. Note that f_x is an extendable plf on A and that $v(f_x) \leqslant f(x^*x)$ (c.f. [2, Lemma 37.6]).

DEFINITION 2.16. One plf f on a topological $*$-algebra A is said to *dominate* another plf g on A if f–g is a plf on A or, equivalently, $g(x^*x) \leqslant f(x^*x)$, for all $x \in A$.

3. MAIN RESULTS

The starting point for our results is the following theorem of Ng and Warner [15, Theorem 4].

THEOREM 3.1. *Every extendable* plf *on a complete metrizable topological $*$-algebra is continuous.*

This result can be extended to a class of LMC $*$-algebras via a structure theorem due to Akkar [1]. To put this structure theorem in perspective, we recall the well-known result [10, Proposition 3.7.5] that every quasi-complete (hence, sequentially complete) bornological locally convex space is the inductive limit of a family of Banach spaces.

THEOREM 3.2. *If A is a sequentially complete bornological* LMC *$*$-algebra, then every extendable* plf *on A is continuous.*

Proof. The structure theorem referred to above essentially states that A is the inductive liit of a family of Fréchet $*$-algebras; the key idea in Akkar's proof is an argument due to Dixon and Fremlin [8] using Garling's completeness theorem [9, Theorem 1]. Now, if f is an extendable plf on A, then its restriction to each of the Fréchet $*$-algebras in the inductive family is also an extendable plf. Consequently, each of these restrictions is continuous by Theorem 3.1 and, by a general property of inductive limits [10, Proposition 2.12.1], it follows that f is continuous on A. ∎

Of course, Theorem 3.2 extends to locally convex $*$-algebras which are inductive limits of families of either complete metrizable locally convex $*$-algebras or sequentially complete bornological LMC $*$-algebras. An alternate proof of Theorem 3.2 can be given by using the Dixon–Fremlin argument [8], together with Theorem 3.1, to prove that every extendable plf on a sequentially complete LMC $*$-algebra is necessarily bounded. Theorem 3.2 then follows by noting that, on bornological locally convex spaces, bounded linear functionals are continuous. It should be noted that, although sequential completeness is more general than quasi-completeness for arbitrary locally convex spaces [18, Exercise 21, p. 195], the Dixon–Fremlin construction shows that the two notions coincide for LMC algebras.

Next, we give a version of Ford's square root lemma which applies in the setting of complete LMC $*$-algebras.

LEMMA 3.3. *Let A be a complete* LMC *$*$-algebra. If $x = x^*$ and* $\mathrm{r}(x, A) < 1$, *then there exists a unique $y = y^*$ in A such that $x = y \circ y = 2y - y^2$ and* $\mathrm{r}(y, A) < 1$.

Proof. First, in Notation 2.5, $A = \operatorname{proj}\lim B_\lambda$ (Theorem 2.6) and $\mathrm{r}(x, A) = \sup\{\mathrm{r}(x_\lambda, B_\lambda): \lambda \in \Lambda\}$ (Theorem 2.8). Let $\mathrm{r}(x, A) = \eta < 1$; then, for each $\lambda \in \Lambda$, $\mathrm{r}(x_\lambda, A) \leqslant \eta < 1$. Hence, by Ford's square root lemma [2, Propositions 8.13, 12.11] (applied to x_λ in the Banach $*$-algebra B_λ), there exists a unique $y_\lambda = y_\lambda^*$ such that $x_\lambda = y_\lambda \circ y_\lambda$ and $\mathrm{r}(y_\lambda, B_\lambda) \leqslant \eta$. Now, suppose λ, $\mu \in \Lambda$ and $\lambda < \mu$; then, $z_\lambda = \pi_{\lambda\mu}(y_\mu)$ satisfies $z_\lambda = z_\lambda^*$, $x_\lambda = z_\lambda \circ z_\lambda$. Also, $\mathrm{r}(z_\lambda, B_\lambda) = \lim_{n\to\infty} \sqrt[n]{p_\lambda(z_\lambda^n)}$ and $p_\lambda(z_\lambda^n) = p_\lambda(\pi_{\lambda\mu}(y_\mu^n)) \leqslant p_\mu(y_\mu^n)$, so $\mathrm{r}(z_\lambda, B_\lambda) \leqslant \lim_{n\to\infty} \sqrt[n]{p_\mu(y_\mu^n)} = \mathrm{r}(y_\mu, B_\mu) \leqslant \eta < 1$. Consequently, by uniqueness, $y_\lambda = z_\lambda = \pi_{\lambda\mu}(y_\mu)$. Therefore (Theorem 2.6), there exists a unique y in A such that $\pi_\lambda(y) = y_\lambda$, for all $\lambda \in \Lambda$. It is clear that $y^* = y$, $x = y \circ y$, and $\mathrm{r}(y, A) = \sup\{\mathrm{r}(y_\lambda, B_\lambda)\colon \lambda \in \Lambda\} \leqslant \eta < 1$. Finally, if $z \in A$ is such that $z^* = z$, $x = z \circ z$, and $\mathrm{r}(z, A) < 1$, then, for each $\lambda \in \Lambda$, $z_\lambda^* = z_\lambda$, $x_\lambda = z_\lambda \circ z_\lambda$, $\mathrm{r}(z_\lambda, B_\lambda) < 1$, and so the uniqueness in Ford's square root lemma on B_λ implies the $z_\lambda = y_\lambda$; hence, $z = y$. ∎

Now, we state a generalization of the main result of Murphy [14, Theorem]. The proof given in [14], which relies upon the principle of uniform boundedness (and so is valid for our setting as well), is omitted.

LEMMA 3.4 (Murphy). *Let A be a barreled locally convex $*$-algebra which factors. If every nonzero* plf *on A dominates a nonzero continuous* plf *on A, then every* plf *on A is continuous.*

Using Lemmas 3.3 and 3.4, an extension of Murphy's automatic-continuity result for plfs on factoring commutative Banach $*$-algebras [14, Corollary] can be obtained; the essential ingredients in the proof remain the same with the exception of spectral boundedness.

THEOREM 3.5. *Let A be a complete commutative* LMC *$*$-algebra which factors and has the following properties*:

(a) *A is barreled.*

(b) *Every extendable* plf *on A is continuous.*

(c) *A is almost spectrally bounded.*

Then every plf *on A is continuous.*

Remark 3.6. Observe that properties (a) and (b) hold if A is Fréchet or bornological, while property (c) holds if A is a Q-algebra.

Proof of Theorem 3.5. Let f be an arbitrary nonzero plf on A; by property (a) and Lemma 3.4, it suffices to find a nonzero continuous plf on A that is dominated by f. For each $u \in A$, f_u (see Notation 2.15) is an extendable plf on A and, as such, is continuous by property (b). Suppose that $f_u = 0$, for all $u \in A$. Then

$$|f(u^*yz)|^2 \leqslant f(u^*y^*yu)\, f(z^*z) = f_u(y^*y)\, f(z^*z) = 0,$$

for all $u, y, z \in A$; that is, $f(A^3) = 0$ which cannot be, since $A^3 = A$. Consequently, there exists some $u \in A$ such that $f_u \neq 0$. However, by property (c), this means that there exists a spectrally bounded x (chosen without loss of generality so that $r(x^*x, A) < 1$) such that $f_u(x^*x) > 0$. By commutativity of A, it follows that f_x is a nonzero continuous plf on A. Now, by Lemma 3.3, there exists a $y \in A$, $y = y^*$, such that $x^*x = 2y - y^2$. Thus, for each $z \in A$,

$$\begin{aligned}(f - f_x)(z^*z) &= f(z^*z - x^*z^*zx) = f(z^*z - z^*x^*xz) \\ &= f(z^*z - 2z^*yz + z^*y^2 - z) \\ &= f((z^* - z^*y)(z - yz)) = f((z - yz)^* (z - yz)) \geqslant 0,\end{aligned}$$

and so f dominates f_x. ∎

Our final theorem in this section uses the existence of a pointwise operator bounded approximate identity (pobai, see Definition 2.11) in the algebra to obtain automatic continuity of plfs on the algebra. The result is new even for Banach $*$-algebras.

THEOREM 3.7. *Let A be a barreled locally convex $*$-algebra which factors. Suppose that every extendable* plf *on A is continuous. Then*

(a) *if A has a one-sided* pobai $\{e_\omega: \omega \in \Omega\}$, *then every trace on A is continuous.*

(b) *if A has a central* pobai $\{e_\omega: \omega \in \Omega\}$, *then every* plf *on A is continuous.*

Proof. Let f be a plf on A. For each $a \in A, f_a$ is an extendable plf and so, is continuous; moreover, from the polarization identity

$$4xay = \sum_{k=0}^{3} i^k(y + i^k x^*)^* a(y + i^k x),$$

it follows immediately that the linear functional $a \to f(xay)$ is continuous on A, for all $x, y \in A$. Next, without loss of generality (using $\{e_\omega^*\}$ in place of $\{e_\omega\}$, if necessary), assume that $\{e_\omega : \omega \in \Omega\}$ is a right pobai for A.

Let $a \in A = A^2 = A^4$ be given; then $a = \sum_{j=1}^{m} w_j x_j y_j z_j$. If f is a trace on A (part (a) of the theorem), then

$$\begin{aligned}f_{e_\omega}(a) &= f(e_\omega^* a e_\omega) = f(a e_\omega e_\omega^*) \\ &= \sum_{j=1}^{m} f(w_j x_j y_j z_j e_\omega e_\omega^*) \\ &= \sum_{j=1}^{m} f(y_j(z_j e_\omega)(e_\omega^* w_j) x_j),\end{aligned}$$

whereas if $\{e_\omega\}$ is central (part (b) of the theorem), then

$$f_{e_\omega}(a) = f(e_\omega^* a e_\omega) = \sum_{j=1}^{m} f(e_\omega^* w_j x_j y_j z_j e_\omega)$$

$$= \sum_{j=1}^{m} f(w_j(e_\omega^* x_j)(y_j e_\omega) z_j).$$

In either case, it follows that

$$f(a) = \lim_\omega f_{e_\omega}(a),$$

every $a \in A$. Using the expressions above for $f_{e_\omega}(a)$, and the fact that $\{e_\omega: \omega \in \Omega\}$ is right pobai, it is easily shown that $\{f_{e_\omega}(a): \omega \in \Omega\}$ is bounded in C; equivalently, $\{f_{e_\omega}: \omega \in \Omega\}$ is a $\sigma(A', A)$-bounded subset of the topological dual A' of A. The Banach–Steinhaus theorem [10, Proposition 3.6.5] thus implies that f is continuous. ∎

4. Applications

The results of the previous section have application to abstract harmonic analysis in the nature of automatic continuity of plfs on group algebras composed of test functions. Our principal results to date concern locally compact Abelian (LCA) groups and locally compact groups with compact connected component. For G an arbitrary LCA group, let $S(G)$ denote the Schwartz–Bruhat test function space of rapidly decreasing complex-valued regular functions on G, regarded as a locally convex $*$-algebra with respect to the operations of convolution and involution that it inherits from $L^1(G)$. For the definition and properties of $S(G)$, see [19] for $G = R^n$ and [4] for G arbitrary LCA; an alternate characterization of $S(G)$ may be found in [16]. The members of the topological dual space $S'(G)$ of $S(G)$ are called tempered distributions on G.

Theorem 4.1. *For G an arbitrary* LCA *group, every* plf *on $S(G)$ is continuous (i.e., is a tempered distribution on G).*

Proof. First consider the case where G is an elementary Lie group (i.e., $G = R^n \times T^m \times Z^k \times F$, F a finite Abelian group). In this case, $S(G)$ is a Fréchet space; hence, every extendable plf on $S(G)$ is continuous by Theorem 3.1. Further, the work of Dixmier and Malliavin [7], together with standard properties of $S(G)$ (to reduce to the case $G = R^p$), shows that $S(G)$ factors. Since $S(G)$ has a sequential central approximate identity (c.f. [11, Theorem II.1.6, p. 127]), Theorem 3.7(b) thus implies that every plf on $S(G)$ is continuous.

If G is an arbitrary LCA group, then $S(G) = \text{ind lim } S(H/K)$ (inductive limit), where the limit is taken over pairs (H, K) of subgroups of G such that H is open and compactly generated, K is a compact subgroup of H, and H/K is a Lie group (necessarily elementary). Standard arguments reveal that if T is a plf on $S(G)$, then T is a plf on $S(H/K)$, for every pair (H, K) as above. Consequently, T is continuous on each $S(H/K)$ and, by a general property of inductive limits [10, Proposition 2,.12.1], it follows that T is continuous on $S(G)$. ∎

Observe that, for a general LCA group G, this continuity of plfs result implies that $S(G)$ factors. For, if $S(G)^2 \subsetneq S(G)$, then by Zorn's lemma there would exists a nonzero linear functional T on $S(G)$ which vanishes on $S(G)^2$. Such a functional T is clearly a plf on $S(G)$; however, since $S(G)^2$ is dense in $S(G)$, T is necessarily discontinuous, which is a contradiction.

Our results in the compact connected component case involve the Schwartz–Bruhat space $D(G)$ of complex-valued compactly supported regular functions on G, whose topological dual $D'(G)$ is the space of distributions on G. The reader is referred to [4 or 11] for the definition and properties of $D(G)$, for G an arbitrary locally compact group. In this general setting, $D(G)$ is a complete barreled locally convex $*$-algebra with respect to convolution and involution inherited from $L^1(G)$; further, factorization holds in every case.

THEOREM 4.2. *If G is an arbitrary locally compact group, then $D(G)$ factors.*

For G a Lie group, Theorem 4.2 is due to Dixmier and Malliavin [7]. The general case follows by using the structure theory of locally compact groups and properties of $D(G)$, including the following facts:

(1) If G is locally compact and $G = \text{proj lim } G_\lambda$, G_λ locally compact, then $D(G)$ factors $\Leftrightarrow$ each $D(G_\lambda)$ factors.

(2) For G locally compact, $D(G)$ factors $\Leftrightarrow$ for every open subgroup H of G, $D(H)$ factors $\Leftrightarrow$ for one open subgroup H of G, $D(H)$ factors. The details of the proof are left to the reader.

THEOREM 4.3. *For G an arbitrary compact group, every* plf *on $D(G)$ is continuous (i.e., is a distribution on G).*

Proof. The idea of the proof is like that of Theorem 4.1. First assume that G is a compact Lie group. Then $D(G)$ is a Frèchet space, so every extendable plf on $D(G)$ is continuous. Moreover, $D(G)$ factors and, if $\{\phi_\omega\}$ is an approximate identity for $D(G)$ (see [11, Theorem II.1.6]), then it follows easily that $\{\psi_\omega\}$, defined by

$$\psi_\omega(x) = \int_G \phi_\omega(y^{-1}xy)\, dy, \qquad x \in G,$$

(where dy denotes normalized Haar measure on G) is a central approximate identity for $D(G)$. Hence, every plf on $D(G)$ is continuous.

If G is an arbitrary compact group, then G is Lie projective (i.e., $G = \operatorname{proj}\lim G_\lambda$, where each G_λ is a compact Lie group) and so $D(G) = \operatorname{ind}\lim D(G_\lambda)$. Since each $G_\lambda \cong G/K_\lambda$, where K_λ is a compact normal subgroup of G, it follows that, if T is a plf on $D(G)$, then T is a plf on each $D(G_\lambda)$ and, as such, is continuous on $D(G_\lambda)$; consequently, T is continuous on $D(G)$. ∎

Remark 4.4. There is a second proof of Theorem 4.3 along the lines of Varopoulos' proof of the continuity of plfs on Banach $*$-algebras with bounded tais. It uses the strong factorization result [7, Remark 4.10] (extended as in Theorem 4.2 to an arbitrary compact group G) that, if B_1 is a closed bounded (equivalently, compact) subset of $D(G)$, then there exist ϕ, $\psi \in D(G)$ and B_2 a closed bounded subset of $D(G)$ such that $B_1 = \phi * B_2 * \psi$. Because every extendable plf on $D(G)$ is continuous, polarization implies that for a general plf T on $D(G)$, the linear functional $\rho \to \langle \phi * \rho * \psi, T\rangle$ is continuous on $D(G)$. Thus, $\langle B_1, T\rangle = \langle \phi * B_2 * \psi, T\rangle$ is bounded in C and, since $D(G)$ is bornological, T is continuous on $D(G)$.

THEOREM 4.5. *If G is a locally compact group containing an open subgroup H such that every plf on $D(H)$ is continuous, then every plf on $D(G)$ is continuous.*

Proof. Let T be a plf on $D(G)$; then (by restriction) T is a plf on $D(H)$ and, as such, is continuous. Therefore, the seminorm $p_T(\psi) = \langle \psi^* * \psi, T\rangle^{1/2}$ is continuous on $D(H)$ [11, Proposition III.2.1]. Fix $x \in G$, and consider the subspace $D(xH) = \{\phi \in D(G): \operatorname{supp}(\phi) \subseteq xH\} = \delta_x * D(H)$, where $(\delta_x * \psi)(y) = \psi(x^{-1}y)$. If $\phi_n \to 0$ in $D(xH)$, then there exist $\psi_n \to 0$ in $D(H)$ such that $\phi_n = \delta_x * \psi_n$, for all n. Therefore, $p_T(\phi_n) = \langle \phi_n^* * \phi_n, T\rangle^{1/2} = \langle \psi_n^* * \psi_n, T\rangle^{1/2} = p_T(\psi_n) \to 0$, and so p_T is continuous on $D(xH)$. Since x is arbitrary, it follows that the seminorm p_T is continuous on $D(G)$. Consequently, the sesquilinear form b: $D(G) \times D(G) \to C$, defined by $b(\phi, \psi) = \langle \psi^* * \phi, T\rangle$ is separately continuous (indeed, $|b(\phi, \psi)| \leqslant p_T(\phi)\, p_T(\psi)$). Further, b is clearly adjunctive (i.e., $b(\rho * \phi, \psi) = b(\phi, \rho^* * \psi)$, for all $\phi, \psi, \rho \in D(G)$; hence, by the Kernel theorem [11, Theorem II.3.6], there exists a distribution R in $D'(G)$ such that $\langle \psi^* * \phi, R\rangle = b(\phi, \psi) = \langle \psi^* * \phi, T\rangle$, for all $\phi, \psi \in D(G)$. Thus, $T = R$ on $D(G)^2 = D(G)$, implying $T \in D'(G)$. ∎

COROLLARY 4.6. *If G is a locally compact group with compact connected component G_0 of the identity, then every* plf *on $D(G)$ is continuous.*

Proof. Every locally compact group G contains an open subgroup H such that H/G_0 is compact. If G_0 is compact, then H is compact, so Theorem 4.3 applies, and the corollary then follows from Theorem 4.5. ∎

Remark 4.7. The continuity of extendable plfs on a topological $*$-algebra A is related to the continuity of $*$-representations of A on Hilbert space. For example, if A is a barreled locally convex $*$-algebra such that every extendable plf on A is continuous, then every $*$-representation π: $A \to B(H_\pi)$ of A on Hilbert space is continuous. This applies to the cases $A = S(G)$, G an LCA group, and $A = D(G)$, G a locally compact group with compact G_0; thus, $S(G)$ and $D(G)$ are true group algebras in these settings.

We sketch the proof. First, for each $h \in H_\pi$, the linear functional $a \to (\pi(a)\, h \mid h)$ is an extendable plf on A, and so is continuous. By polarization, it follows that the linear functional $a \to (\pi(a)\, h \mid k)$ is continuous on A, for all $h, k \in H_\pi$; that is, the *linear map* π: $A \to (B(H_\pi), \tau_{\mathrm{wo}})$ is continuous (τ_{wo} is the weak operator topology on $B(H_\pi)$). Consequently, π has closed graph in $A \times (B(H_\pi), \tau_{\mathrm{wo}})$; hence, a fortiori π has closed graph in $A \times B(H_\pi)$, and since A is barreled, $\pi: A \to B(H_\pi)$ is continuous [10, Theorem 3.17.4].

REFERENCES

1. M. AKKAR, Sur la structure des algèbres topologiques localement multiplicativement convexes, *C. R. Acad. Sci. Paris* **279** (1974), 941–944.
2. F. F. BONSALL AND J. DUNCAN, "Complete Normed Algebras," Springer–Verlag, Berlin/New York, 1973.
3. R. M. BROOKS, On locally m-convex $*$-algebras, *Pacific J. Math.* **23** (1967), 5–23.
4. F. BRUHAT, Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques, *Bull. Soc. Math. France* **89** (1961), 43–75.
5. P. J. COHEN, Factorization in group algebras, *Duke Math. J.* **26** (1959), 199–205.
6. I. G. CRAW, Factorisation in Fréchet algebras, *J. London Math. Soc.* **44** (1969), 607–611.
7. J. DIXMIER AND P. MALLIAVIN, Factorisations de functions et de vecteurs indéfiniment differentiables, *Bull. Sci. Math. Ser.* 2 **102** (1978), 305–330.
8. P. G. DIXON AND D. H. FREMLIN, A remark concerning multiplicative functionals on *LMC* algebras, *J. London Math. Soc.* **5**(2) (1972), 231–232.
9. D. J. H. GARLING, On topological sequence spaces, *Math. Proc. Cambridge Philos. Soc.* **63** (1967), 997–1019.
10. J. HORVÁTH, "Topological Vector Spaces and Distributions," Vol. I, Addison–Wesley, Reading, Mass., 1966.
11. D. L. JOHNSON, "The Theory of Distributions on Locally Compact Groups with Applications to Group Representations," Ph.D. Dissertation, Univ. of Minnesota, 1976.
12. E. KILLAM, The spectrum and the radical in locally m-convex algebras, *Pacific J. Math.* **12** (1962), 581–588.

13. E. A. MICHAEL, Locally multiplicatively convex topological algebras, *Mem. Amer. Math. Soc.* **11** (1952).
14. I. S. MURPHY, Continuity of positive linear functionals on Banach $*$-algebras. *Bull. London Math. Soc.* **1** (1969), 171–173.
15. S.-B. NG AND S. WARNER, Continuity of positive and multiplicative functionals, *Duke Math. J.* **39** (1972), 281–284.
16. M. SCOTT OSBORNE, On the Schwartz–Bruhat space and the Paley–Wiener theorem for locally compact Abelian groups, *J. Funct. Anal.* **19** (1975), 40–49.
17. J.-L. OVAERT, Factorisation dans les algèbres et modules de convolution, *C. R. Acad. Sci. Paris* **265** (1967), 534–535.
18. H. H. SCHAEFER, "Topological Vector Spaces," Springer–Verlag, New York/Berlin, 1971.
19. L. SCHWARTZ, "Théorie des Distributions," new Ed., Hermann, Paris, 1966.
20. N. T. VAROPOULOS, Sur les formes positives d'une algèbre de Banach, *C. R. Acad. Sci. Paris* **258** (1964), 2465–2467.